$\sup + \inf$ for Riemannian surfaces and $\sup \times \inf$ for bounded domains of $\mathbb{R}^n$, $n \geq 3$

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Abstract

On a Riemannian surface, we give a condition to obtain a minoration of $\sup + \inf$. On an open bounded set of $\mathbb{R}^n$ ($n \geq 3$) with smooth boundary, we have a minoration of $\sup \times \inf$ for prescribed scalar curvature equation with Dirichlet condition.

Keywords: Riemannian Surface, $\sup + \inf$, $\sup \times \inf$, Dirichlet condition.

In this paper, we study some inequalities of type $\sup + \inf$ (in dimension 2) and $\sup \times \inf$ (in dimension $n \geq 3$). We denote $\Delta = -\nabla_i(\nabla^i)$ the geometric laplacian.

The paper is linking to the Note presented in Comptes Rendus de l'Académie des Sciences de Paris (see [B1])

In dimension 2, we work on Riemannian surface (M, g) and we consider the following equation:

$$\Delta u + f = V e^u \qquad (E_1)$$

where f and V are two functions.

We are going to prove a minoration of $\sup u + \inf u$ under some conditions on f and V.

Where $f = R$, with R the scalar curvature of M, we have the scalar curvature equation studied by T. Aubin, H. Brezis, YY. Li, L. Nirenberg, R. Schoen.

In the case $f = R = 2\pi$ and $M = \mathbb{S}^2$, we have a lower bound for $\sup + \inf$ assuming V non negative, bounded above by a positive constant b and without condition on ∇V (see Bahoura [B]).

The problem was studied when we suppose $V = V_i$ uniformly lipschitzian and between two positive constants. (See Bahoura [B] and Li [L]). In fact, there exists $c = c(a, b, A, M)$ such that for all sequences u_i and V_i satisfying:

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$$\Delta u_i + R = V_i e^{u_i},\ 0 < a \leq V_i(x) \leq b \text{ and } ||\nabla V_i||_\infty \leq A,$$

we have,

$$\sup_M u_i + \inf_M u_i \geq c \ \forall\ i.$$

We have some results about L^∞ boundness and asymptotic behavior for the solutions of euqations of this type on open set of $\mathbb{R}^2$, see [BM], [S], [SN 1] and [SN 2].

Here, we try to study the same problem with minimal conditions on f and V, we suppose $0 \leq V \leq b$ and without assumption on ∇V.

Theorem 1. Assume (M, g) *a Riemannian surface and* f, V *two functions satisfying:*

$$f(x) \geq 0, \text{ and } 0 \leq V(x) \leq b < +\infty,\ \ \forall\ x \in M.$$

suppose u *solution of:*

$$\Delta u + f = V e^u.$$

then:
if $0 < \int_M f \leq 8\pi$, *there exists a constant* $c = c(b, f, M)$ *such that:*

$$\sup_M u + \inf_M u \geq c,$$

if $8\pi < \int_M f < 16\pi$, *there exists* $C = C(f, M) \in]0, 1[$ *and* $c = c(b, f, M)$ *such that:*

$$\sup_M u + C \inf_M u \geq c.$$

Remark: In fact, we can suppose $f \equiv k$ a constant. (See [B1]).

Now, we work on a smooth bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$.

Let us consider the following equations:

$$\Delta u_\epsilon = {u_\epsilon}^{N-1-\epsilon},\ \ u_\epsilon > 0 \ \text{ in } \ \Omega \ \text{ and } \ u_\epsilon = 0 \ \text{ on } \partial\Omega \qquad (E_2).$$

with $\epsilon \geq 0$, $N = \dfrac{2n}{n-2}$.

The existence result for those equations depends on the geometry of the domain. For example, if we suppose, Ω starshaped and $\epsilon = 0$, the Pohozaev identity assure a nonexistence result. If $\epsilon = 0$, under assumption on Ω, we can have an existence result. When $\epsilon > 0$ there exists a solutions for the previous equation.

For $\epsilon > 0$, [AP], [BP] and [H] , studied some properties of the previous equation.

On unit ball of $\mathbb{R}^n$, Atkinson-Peletier(see [AP]) have proved:

$$\lim_{\epsilon \to 0} [\sup_{B_1(0)} u_\epsilon \inf_{B_k(0)} u_\epsilon] = \left(\frac{1}{|k|} - 1 \right),$$

with $|x| = k < 1$.

In [H], Z-C Han, has proved the same estimation on a smooth open set $\Omega \subset \mathbb{R}^n$ with the following condition :

$$\lim_{\epsilon \to 0} \frac{\int_\Omega |\nabla u_\epsilon|^2}{[||u_\epsilon||_{L^{N-1-\epsilon}}]^2} = S_n \qquad (1),$$

with $S_n = \pi n(n-2) \left[\frac{\Gamma(n/2)}{\Gamma(n)} \right]$ the best constant in the Sobolev imbedding.

In fact, the result of Z-C Han (see [H]), is (with (1)),

$$\lim_{\epsilon \to 0} ||u_\epsilon||_{L^\infty} u_\epsilon(x) = \sigma_n (n-2) G(x, x_0), \ \text{ with, } \ x \in \Omega - \{x_0\}.$$

where $x_0 \in \Omega$ and G is the Green function with Dirichlet condition.

In our work, we search to know if it is possible to have a lower bound of $\sup \times \inf$, without the assumption (1).

Theorem 2. For all compact K of Ω, there exists a positive constant $c = c(K, \Omega, n) > 0$, such that for all solution u_ϵ of (E_2) with $\epsilon \in]0, \frac{2}{n-2}]$, we have:

$$\sup_\Omega u_\epsilon \times \inf_K u_\epsilon \geq c.$$

Next, we are intersting by the following equation:

$$\Delta u = u^{N-1} + \epsilon u, \ \ u > 0, \ \text{ in } \ \Omega, \ \text{ and } \ u = 0 \ \text{ on } \ \partial \Omega.$$

We know that in dimension 3, there is no radial solution for the previous equation if $\epsilon \leq \lambda_*$ with $\lambda_* > 0$, see [B N]. Next, we consider $n \geq 4$.

We set G the Green function of the laplacian with Dirichlet condition. For $0 < \alpha < 1$, we denote:

$$\beta = \frac{\alpha}{\sup_\Omega \int_\Omega G(x, y) dy}.$$

Assume $n \geq 4$, we have:

Theorem 3. For all compact K of Ω and all $0 < \alpha < 1$ there is a positive constant $c = c(\alpha, K, \Omega, n)$, such that for all sequences $(\epsilon_i)_{i\in\mathbb{N}}$ with $0 < \epsilon_i \leq \beta$ and $(u_{\epsilon_i})_{i\in\mathbb{N}}$ satisfying:

$$\Delta u_{\epsilon_i} = {u_{\epsilon_i}}^{N-1} + \epsilon_i u_{\epsilon_i}, \ \ u_{\epsilon_i} > 0 \ \text{ and } \ u_{\epsilon_i} = 0 \ \text{ on } \ \partial\Omega, \ \ \forall \ i,$$

we have:

$$\forall \ i, \ \ \sup_{\Omega} u_{\epsilon_i} \times \inf_K u_{\epsilon_i} \geq c.$$

Proof of the Theorem 1:

First part ($0 < \int_M f < 8\pi$) :

We have:

$$\Delta u + f = Ve^u,$$

We multiply by u the previous equation and we integrate by part, we obtain:

$$\int_M |\nabla u|^2 + \int_M fu = \int_M Ve^u u,$$

But $V \geq 0$ and $f \geq 0$, then :

$$\int_M |\nabla u|^2 + \inf_M u \int_M f \leq \sup_M u \int_M Ve^u.$$

On Riemannian surface, we have the following Sobolev inequality, (see [DJLW], [F]):

$$\exists\, C = C(M,g) > 0,\ \forall v \in H_1^2(M),\ \log\left(\int_M e^v\right) \leq \frac{1}{16\pi}\int_M |\nabla v|^2 + \frac{1}{Vol(M)}\int_M v + \log C.$$

Let us consider G the Green function of the laplacian such that:

$$G(x,y) \geq 0 \ \text{ and, } \int_M G(x,y)dV_g(y) \equiv k = \text{constant}.$$

Then,

$$u(x) = \frac{1}{Vol(M)}\int_M u + \int_M G(x,y)[V(y)e^{u(y)} - f(y)]dV_g(y),$$

and,

$$\inf_M u = u(x_0) \geq \frac{1}{Vol(M)}\int_M u - C_1,$$

with,

$$\int_M [G(x_0,y)f(y)] \leq \sup_M f \int_M G(x_0,y)dV_g(y) = k \sup_M f = C_1.$$

But, $\int_M Ve^u = \int_M f > 0$, we obtain,

$$\left(\int_M f\right)(\sup_M u + \inf_M u) \geq -2C_1\int_M f + \frac{2}{Vol(M)}\left(\int_M u\right)\left(\int_M f\right) + \int_M |\nabla u|^2,$$

thus,

$$\sup_M u + \inf_M u \geq 2\left[\frac{1}{Vol(M)}\int_M u + \frac{1}{2\int_M f}\int_M |\nabla u|^2\right] - 2C_1.$$

If we suppose, $0<\int_M f \leq 8\pi$, we obtain $\dfrac{1}{2\int_M u} \geq \dfrac{1}{16\pi}$ and then:

$$\sup_M u + \inf_M u \geq 2\left[\frac{1}{Vol(M)}\int_M u + \frac{1}{16\pi}\int_M |\nabla u|^2\right] - 2C_1,$$

We use the previous Sobolev inequality, we have:

$$\sup_M u + \inf_M u \geq -2C_1 - 2\log C + 2\log\left(\int_M e^u\right),$$

but,

$$\int_M f = \int_M Ve^u \leq b\int_M e^u,$$

then,

$$\int_M e^u \geq \frac{1}{b}\int_M f,$$

and finaly,

$$\sup_M u + \inf_M u \geq -2C_1 - 2\log C + 2\log\left(\frac{1}{b}\int_M f\right).$$

Second part ($8\pi < \int_M f < 16\pi$):

Like en the first part, we have:

$$a)\ \ \int_M |\nabla u|^2 + \inf_M u \int_M f \leq \sup u \int_M f,$$

$$b)\ \ \log\left(\int_M e^u\right) \leq \frac{1}{16\pi}\int_M |\nabla u|^2 + \frac{1}{Vol(M)}\int_M u + \log C,$$

$$c)\ \ \inf_M u \geq \frac{1}{Vol(M)}\int_M u - C_1.$$

We set $\lambda > 0$. We use $a), b), c)$ and we obtain:

$$\left(\int_M f\right)(\sup_M u + \lambda \inf_M u) \geq -(\lambda+1)C_1\int_M f + \frac{(1+\lambda)}{Vol(M)}\left(\int_M u\right)\left(\int_M f\right) + \int_M |\nabla u|^2,$$

thus,

$$\sup_M u + \lambda\inf_M u \geq -(\lambda+1)C_1 + (1+\lambda)\left[\frac{1}{Vol(M)}\int_M u + \frac{1}{(1+\lambda)\int_M f}\int_M |\nabla u|^2\right].$$

We choose $\lambda > 0$, such that, $\dfrac{1}{(1+\lambda)\int_M f} \geq \dfrac{1}{16\pi}$,

thus, $(1+\lambda)\int_M f \leq 16\pi$, $0<\lambda \leq \dfrac{16\pi - \int_M f}{\int_M f} < 1$.

Finaly, the choice of λ, give:

$$\sup_M u + \lambda \inf_M u \geq -(\lambda+1)C_1 - (1+\lambda)\log C + (1+\lambda)\log\left(\frac{1}{b}\int_M f\right).$$

If we take $\lambda = \dfrac{16\pi - \int_M f}{\int_M f} \in]0,1[$, we obtain:

$$\sup_M u + \left(\frac{16\pi - \int_M f}{\int_M f}\right)\inf_M u \geq -C_1 \frac{16\pi}{\int_M f} - \frac{16\pi}{\int_M f}\log C + \frac{16\pi}{\int_M f}\log\left(\frac{1}{b}\int_M f\right).$$

Proof of theorems 2 and 3:

Here, we give two methods to prove the theorems 2 and 3, but we do the proof only for the theorem 2. In the first proof we use the Moser iterate scheme, the second proof is direct.

Method 1: by the Moser iterate scheme.

We argue by contradiction and we suppose:
$\exists\ K \subset\subset \Omega,\ \forall\, c>0,\ \exists\, \epsilon_c \in]0, \dfrac{2}{n-2}]$ such that:

$$\Delta u_{\epsilon_c} = {u_{\epsilon_c}}^{N-1-\epsilon_c},\ u_{\epsilon_c} > 0 \text{ in } \Omega \text{ and } u_{\epsilon_c} = 0 \text{ on } \partial\Omega,$$

with,

$$\sup_\Omega u_{\epsilon_c} \times \inf_K u_{\epsilon_c} \leq c$$

We take $c = \dfrac{1}{i}$, there exists a sequence $(\epsilon_i)_{i\geq 0}$, such that $\forall\ i \in \mathbb{N}, \epsilon_i \in]0, \dfrac{n}{n-2}]$ and

$$\Delta u_{\epsilon_i} = {u_{\epsilon_i}}^{N-1-\epsilon_i},\ u_{\epsilon_i} > 0 \text{ in } \Omega \text{ and } u_{\epsilon_i} = 0, \text{ on } \partial\Omega\ (*)$$

with,

$$\sup_\Omega u_{\epsilon_i} \times \inf_K u_{\epsilon_i} \leq \frac{1}{i} \to 0\ (**).$$

Clearly the function u_{ϵ_i} which satisfy $(*)$, there exists $x_{\epsilon_i} \in \Omega$ such that:

$$\sup_\Omega u_{\epsilon_i} = \max_\Omega u_{\epsilon_i} = u_{\epsilon_i}(x_{\epsilon_i}).$$

Lemma:

There exists $\delta = \delta(\Omega, n) > 0$ such that for all $\epsilon > 0$ and $u_\epsilon > 0$, solution of our problem with $x_\epsilon \in \Omega$, $\sup_\Omega u_\epsilon = u_\epsilon(x_\epsilon)$ we have:

$$d(x_\epsilon, \partial\Omega) \geq \delta.$$

Proof of the lemma:

We argue by contradiction. We suppose: $\forall\ \delta > 0, \exists\ x_{\epsilon_{i_\delta}}$ such that: $d(x_{\epsilon_{i_\delta}}, \partial\Omega) \leq \delta$.

We take $\delta = \dfrac{1}{j}, j \to +\infty$, we have a subsequence ϵ_{i_j}, noted ϵ_i, such that, $d(x_{\epsilon_i}, \partial\Omega) \to 0$.

Let us consider G the Green function of the laplacian with Dirichlet condition and w satisfying:

$$\Delta w = 1 \text{ in } \Omega \text{ and } w = 0 \text{ on } \partial\Omega.$$

Using the variational method, we can prove the existence of w and $w \in \mathcal{C}^\infty(\bar{\Omega})$.

The Green representation formula and the fact $x_{\epsilon_i} \to y_0 \in \partial\Omega$ give:

$$0 = w(y_0) \leftarrow w(x_{\epsilon_i}) = \int_\Omega G(x_{\epsilon_i}, y)dy,$$

we can write,

$$\int_\Omega G(x_{\epsilon_i}, y)dy \to 0.$$

The function u_{ϵ_i} satisfy $(*)$ and thus:

$$u_{\epsilon_i}(x_{\epsilon_i}) \leq (\max_\Omega u_{\epsilon_i})^{N-1-\epsilon_i} \int_\Omega G(x_{\epsilon_i}, y)dy,$$

consequently,

$$1 \leq [u_{\epsilon_i}(x_{\epsilon_i})]^{N-2-\epsilon_i} \int_\Omega G(x_{\epsilon_i}, y)dy.$$

Then,

$$u_{\epsilon_i}(x_{\epsilon_i}) \to +\infty \text{ and } x_{\epsilon_i} \to y_0 \in\ \partial\Omega\ \ (***).$$

But, if we use the result of Z-C.Han (see [H] page 164) and [DLN] (pages 44-45 and 50-53) and the moving plane method (see [GNN]) we obtain:

if Ω is smooth bounded domain, f a function in $\mathcal{C}^1$ and u is a solution of:

$$\Delta u = f(u), \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega,$$

there exists two positive constants δ and γ, which depend only on the geometry of the domain Ω, such that:

$\forall\, x \in \{z, d(z,\partial\Omega) \leq \delta\}$, $\exists\, \Gamma_x \subset \{z, d(z,\partial\Omega) \geq \dfrac{\delta}{2}\}$ with $mes(\Gamma_x) \geq \gamma$ et $u(x) \leq u(\xi)$ for all $\xi \in \Gamma_x$.

Thus,

$$u(x) \leq \frac{1}{mes(\Gamma_x)} \int_{\Gamma_x} u \leq \frac{1}{\gamma} \int_{\Omega'} u \qquad (*'),$$

with $\Omega' \subset\subset \Omega$.

If we replace x by x_{ϵ_i}, u by u_{ϵ_i} and we take $\Omega' = \{z \in \Omega, d(z,\partial\Omega) \geq \dfrac{\delta}{2}\}$, we obtain (after using the argument of the first eigenvalue like in [H]):

$$+\infty \leftarrow u_{\epsilon_i}(x_{\epsilon_i}) \leq \frac{1}{\gamma} \int_{\Omega'} u_{\epsilon_i} \leq c_2(\Omega', n) < \infty,$$

it is contradiction. The lemma is proved.

We continue the proof of the Theorem.

Without loss of generality , we can assume $x_{\epsilon_i} \to y_0$. We consider $(x_{\epsilon_i})_{i\geq 0}$ and $\mu > 0$, such that $x_{\epsilon_i} \in B(y_0, \mu) \subset\subset \Omega$. (we take $\mu = \dfrac{\delta}{2}$ for example).

We have:

$$u_{\epsilon_i}(x) = \int_\Omega G(x,y) {u_{\epsilon_i}}^{N-1-\epsilon_i}(y)dy$$

According to the properties of the Green functions and maximum priciple, on the compact K of Ω:

$$G(x,y) \geq c_3 = c(K, \Omega, n) > 0,\ \forall\, x \in K,\ \ y \in B(y_0, \mu).$$

Thus,

$$\inf_K u_{\epsilon_i} = u_{\epsilon_i}(y_{\epsilon_i}) \geq c_3 \int_{B(y_0,\mu)} {u_{\epsilon_i}}^{N-1-\epsilon_i},$$

and then,

$$\int_{B(y_0,\mu)} {u_{\epsilon_i}}^{N-\epsilon_i} \leq (\sup_\Omega u_{\epsilon_i}) \times \int_{B(y_0,\mu)} {u_{\epsilon_i}}^{N-1-\epsilon_i} \leq \frac{(\sup_\Omega u_{\epsilon_i} \times \inf_K u_{\epsilon_i})}{c_3} \to 0.$$

Finaly,

$$0<\int_{B(y_0,\mu)} {u_{\epsilon_i}}^{N-\epsilon_i} \to 0 \qquad (****).$$

Let η be a smooth function such that :

$$0 \leq \eta \leq 1,\ \eta \equiv 1,\ \text{on}\ B(y_0,\mu/2),\ \eta \equiv 0,\ \text{on}\ \Omega - B(y_0, \frac{2\mu}{3}).$$

Set $k>1$. We multiply the equation of u_{ϵ_i} by ${u_{\epsilon_i}}^{2k-1}\eta^2$ and we integrate by part the first member,

$$(2k-1)\int_{B(y_0,2\mu/3)} |\nabla u_{\epsilon_i}|^2 {u_{\epsilon_i}}^{2k-2}\eta^2 + 2\int_{B(y_0,2\mu/3)} <\nabla u_{\epsilon_i}|\nabla \eta> \eta {u_{\epsilon_i}}^{2k-1} = \int_{B(y_0,2\mu/3)} {u_{\epsilon_i}}^{N-2k-2-\epsilon_i}\eta^2,$$

We compute $|\nabla({u_{\epsilon_i}}^k\eta)|^2$ and we deduce:

$$\frac{2k-1}{k^2}\int_{B(y_0,2\mu/3)} |\nabla({u_{\epsilon_i}}^k\eta)|^2 + \frac{2-2k}{k}\int_{B(y_0,2\mu/3)} <\nabla u_{\epsilon_i}|\nabla\eta> {u_{\epsilon_i}}^{2k-1}\eta - \frac{2k-1}{k^2}\int_{B(y_0,2\mu/3)} |\nabla\eta|^2 {u_{\epsilon_i}}^{2k}$$

$$= \int_{B(y_0,2\mu/3)} \eta^2 {u_{\epsilon_i}}^{N+2k-2-\epsilon_i}.$$

And,

$$\int_{B(y_0,2\mu/3)} <\nabla u_{\epsilon_i}|\nabla\eta> {u_{\epsilon_i}}^{2k-1}\eta = \frac{1}{4k}\int_{B(y_0,2\mu/3)} <\nabla({u_{\epsilon_i}}^{2k})|\nabla(\eta^2)> = \frac{1}{4k}\int_{B(y_0,2\mu/3)} \Delta(\eta^2){u_{\epsilon_i}}^{2k}.$$

Then,

$$\frac{2k-1}{k^2}\int_{B(y_0,2\mu/3)} |\nabla({u_{\epsilon_i}}^k\eta)|^2 = \frac{2-2k}{4k^2}\int_{B(y_0,2\mu/3)} \Delta(\eta^2){u_{\epsilon_i}}^{2k} + \frac{2k-1}{k^2}\int_{B(y_0,2\mu/3)} |\nabla\eta|^2 {u_{\epsilon_i}}^{2k} +$$

$$+\int_{B(y_0,2\mu/3)} {u_{\epsilon_i}}^{N+2k-2-\epsilon_i}.$$

But,

$$\int_{B(y_0,2\mu/3)} {u_{\epsilon_i}}^{N+2k-2-\epsilon_i} = \int_{B(y_0,2\mu/3)} ({u_{\epsilon_i}}^{2k}\eta^2)({u_{\epsilon_i}}^{N-2-\epsilon_i}).$$

Using Hölder inequality with $p=(N-\epsilon_i)/2$ and $p'=(N-\epsilon_i)/(N-\epsilon_i-2)$, we obtain:

$$\frac{2k-1}{k^2}[||\nabla(\eta u_{\epsilon_i})||_{L^2(B_0)}]^2 \leq [||u_{\epsilon_i}||_{L^{N-\epsilon_i}(B_0)}]^{N-\epsilon_i-2} \times [||\eta {u_{\epsilon_i}}^k||_{L^{N-\epsilon_i}(B_0)}]^2 + C[||u_{\epsilon_i}||_{L^{2k}(B_0)}]^{2k}$$

with $B_0 = B(y_0, 2\mu/3)$ and $C = C(k,\eta) = \dfrac{2-2k}{4k^2}||\Delta\eta||_\infty + \dfrac{2k-1}{k^2}||\nabla\eta||_\infty$.

Hölder and Sobolev inequalities give,

$$[||\eta {u_{\epsilon_i}}^k||_{L^{N-\epsilon_i}(B_0)}]^2 \leq |B_0|^{2\epsilon_i/[N(N-\epsilon_i)]} K[||\nabla(\eta {u_{\epsilon_i}}^k)||_{L^2(B_0)}]^2.$$

We obtain:

$$\frac{2k-1}{Kk^2|B_0|^{2\epsilon_i/[N(N-\epsilon_i)]}}[||\eta {u_{\epsilon_i}}^k||_{L^{N-\epsilon_i}(B_0)}]^2 \leq [||u_{\epsilon_i}||_{L^{N-\epsilon_i}(B_0)}]^{N-2-\epsilon_i} \times [||\eta {u_{\epsilon_i}}^k||_{L^{N-\epsilon_i}(B_0)}]^2 +$$

$$+C(k,\eta))[||u_{\epsilon_i}||_{L^{2k}(B_0)}]^{2k},$$

with $|B_0| = mes[B(0, 2\mu/3)]$.

We choose $k = \dfrac{N-\epsilon_i}{2}$ and we denote $\alpha_i = [||\eta {u_{\epsilon_i}}^{(N-\epsilon_i)/2}||_{L^{N-\epsilon_i}(B_0)}]^2 > 0$.

We have:

$$c_1\alpha_i \leq \beta_i\alpha_i + c_2\gamma_i,$$

with $c_1 = c_1(N,\mu) > 0, c_2 = c_2(N,\mu) > 0,\ \beta_i = [||u_{\epsilon_i}||_{L^{N-\epsilon_i}}]^{N-2-\epsilon_i}$ and $\gamma_i = [||u_{\epsilon_i}||_{L^{N-\epsilon_i}}]^{N-\epsilon_i}$.

with $\epsilon_i \in]0, \dfrac{2}{n-2}]$. According to $(****)$, we have, $\beta_i \to 0$ and $\gamma_i \to 0$.

Thus,

$$(c_1/2)\alpha_i \leq (c_1-\beta_i)\alpha_i \leq \gamma_i \to 0.$$

Finaly,

$$0 < \int_{B(y_0,\mu/2)} {u_{\epsilon_i}}^{(N-\epsilon_i)^2/2} \leq \int_{B(y_0,2\mu/3)} \eta {u_{\epsilon_i}}^{(N-\epsilon_i)^2/2} \to 0.$$

We iterate this process with $k = \dfrac{(N-\epsilon_i)^2}{4}$ after with $k = \dfrac{(N-\epsilon_i)^r}{2^r}$, $r \in \mathbb{N}^*$, we obtain, for all $q \geq 1$, there exists $l > 0$, such that:

$$\int_{B(y_0,l)} (u_{\epsilon_i})^q \to 0.$$

Using the Green representation formula, we obtain:

$$\forall\ x \in B(x, l'),\ u_{\epsilon_i}(x) = \int_{B(y_0,l)} G(x,y){u_{\epsilon_i}}^{N-1-\epsilon_i}(y)dy + \int_{\partial B(y_0,l)} \partial_\nu G(x,\sigma_l)u_{\epsilon_i}(\sigma_l)d\sigma_l\ \ (*****).$$

where $0 < l' \leq l$.

We have,

$$\int_{B(y_0,l)} {u_{\epsilon_i}}^q = \int_0^l \int_{\partial B(y_0,r)} {u_{\epsilon_i}}^q(r\sigma_r) d\sigma_r dr \to 0,$$

We set, $s_{i,q}(r) = \int_{\partial B(y_0,r)} {u_{\epsilon_i}}^q(r\sigma_r)$. Then,

$$\int_0^l s_{i,q}(r)dr \to 0,$$

We can extract of, $s_{i,q}$, a subsequence which noted $s_{i,q}$ and which tends to 0 almost every-where on $[0,l]$.

First, we choose, $q_1 = \dfrac{q(n+2)}{n-2}$ with $q > \dfrac{n}{2}$, after we choose $l_2 > 0$, such that, $\int_{B(y_0,l_2)} {u_{\epsilon_i}}^{q_1} \to 0$. Finaly, we take $l_1 \in]0, l_2]$, such that, $s_{i,q_1}(l_1) \to 0$. We take $l_0 = \dfrac{l_1}{2} = l'$ in $(*****)$ and $l = l_1$ in $(*****)$, we obtain (if we use Hölder inequality for the two integrals of $(*****)$),

$$\exists\ l_0 > 0, \sup_{B(y_0,l_0)} u_{\epsilon_i} \to 0.$$

But, $x_{\epsilon_i} \to y_0$, for i large, $x_{\epsilon_i} \in B(y_0, l_0)$, which imply,

$$u_{\epsilon_i}(x_{\epsilon_i}) = \max_\Omega u_{\epsilon_i} \to 0.$$

But if we write,

$$u_{\epsilon_i}(x_{\epsilon_i}) = \int_\Omega G(x_{\epsilon_i}, y) {u_{\epsilon_i}}^{N-1-\epsilon_i}(y)dy,$$

we obtain,

$$\max_\Omega u_{\epsilon_i} = u_{\epsilon_i}(x_{\epsilon_i}) \leq (\sup_\Omega u_{\epsilon_i})^{N-1-\epsilon_i} \int_\Omega G(x_{\epsilon_i}, y)dy = [u_{\epsilon_i}(x_{\epsilon_i})]^{N-1-\epsilon_i} w(x_{\epsilon_i}),$$

and finaly,

$$1 \leq u_{\epsilon_i}(x_{\epsilon_i})]^{N-2-\epsilon_i} w(x_{\epsilon_i}).$$

But, $w > 0$ on Ω, $||w||_\infty > 0$ and $N - 2 - \epsilon_i > \dfrac{2}{n-2}$, we have,

$$u_{\epsilon_i}(x_{\epsilon_i}) \geq \frac{1}{[||w||_\infty{}^{1/(N-2-\epsilon_i)}]} \geq c_4(n, \Omega) > 0.$$

It is a contradiction.

For the Theorem 3, we obtain a contradiction if we write:

$$\max_\Omega u_{\epsilon_i} \leq (\max_\Omega u_{\epsilon_i})^{N-1} ||w||_\infty + \max_\Omega u_{\epsilon_i} \epsilon_i \sup_\Omega \int_\Omega G(x,y)dy \leq (\max_\Omega u_{\epsilon_i})^{N-1} ||w||_\infty + \alpha \max_\Omega u_{\epsilon_i},$$

and finaly,

$$\max_\Omega u_{\epsilon_i} \geq \left(\frac{1-\alpha}{||w||_\infty} \right)^{1/(N-2)}.$$

<u>Method 2:</u> proof of theorem 2 directly.

Suppose that:

$$\sup_\Omega \times \inf_K u_i \to 0,$$

then, for $\delta > 0$ small enough, we have:

$$\sup_\Omega u_i \times \inf_{\{x, d(x,\partial\Omega) \geq \delta\}} u_i \to 0.$$

Like in the first method (see [H]), for $\delta > 0$ small,

$$\sup_{\{x, d(x,\partial\Omega) \geq \delta\}} u_i \leq M = M(n, \Omega).$$

We have,

$$u_i(x) = \int_\Omega G(x,y) u_i^{N-1-\epsilon_i} dy.$$

Let us consider K' another compact of Ω, using maximum principle, we obtain:

$$\exists\, c_1 = c_1(K, K', n, \Omega) > 0, \text{ such that } G(x,y) \geq c_1 \ \forall\ x \in K,\ y \in K,$$

thus,

$$\inf_K u_i = u_i(x_i) \geq c_1 \int_{K'} u_i^{N-1-\epsilon_i} dy.$$

We take, $K' = K_\delta = \{x, d(x, \partial\Omega) \geq \delta\}$, there exists $c_2 = c_2(\delta, n, K, \Omega) > 0$ such that:

$$\sup_\Omega u_i \times \inf_K u_i \geq c_2 \int_{K_\delta} u_i^{N-\epsilon_i} dy,$$

we deduce,

$$||u_i||_{N-\epsilon_i}^{N-\epsilon_i} \geq c_2' \sup_\Omega u_i \times \inf_{\{x, d(x,\partial\Omega) \geq \delta\}} u_i + mes(\{x, d(x, \partial\Omega) \leq \delta\}) M^{N-\epsilon_i}.$$

If we take δ small and for i large, we have:

$$||u_i||_{N-\epsilon_i} \to 0.$$

Now, we use the Sobolev imbedding, H_0^1 in L^N, we multiply the equation of u_i by u_i, we intgrate by part and finaly, by Hölder inequality, we obtain:

$$\bar{K}_1||u_i||8N-\epsilon_i{}^2 \leq \bar{K}_2||u_i||_N^2 \leq \int_\Omega |\nabla u_i|^2 = \int_\Omega u_i^{N-\epsilon_i} = ||u_i||_{N-\epsilon_i}^{N-\epsilon_i},$$

we know that, $0 < \epsilon_i \leq \dfrac{2}{n-2}$, the previous inequality:

$$||u_i||_{N-\epsilon_i} \geq \bar{K}_3 > 0, \ \forall \ i,$$

it is a contradiction.

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